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Patterning strategies in early childhood: a mixed methods study examining 3- to 5-year-old children's patterning competencies

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ABSTRACT

Despite increasing data on young children's patterning skills and evidence of a link between early pattern knowledge and later mathematics performance, little is known about the strategies children use to solve patterning tasks. This mixed-methods study examined 3- to 5-year-old children's informal strategies when solving eight patterning tasks with three different repeating patterns. In a sequential qualitative-quantitative design, patterning strategies were first longitudinally observed and categorized into five hierarchically ordered categories (n = 6), and then cross-sectionally tested and quantified (n = 174). Results show that young children use various different patterning strategies, and that both patterning strategies and skills organically develop between the age of three and five. However, despite high correct solution rates, the 5-year olds' strategies still reflect little attention to the patterns' structure, suggesting that a pattern and structure based instruction is needed to help young children abstract the unit of repeat.

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Mathematics is often defined as the science of order, patterns, structures, and logical relationships (Devlin, 2000; Sawyer, 1955; Steen, 1988). Following this view, learning mathematics is developing the ability to recognize patterns, interpret structures, and establish relationships. Accordingly, a growing number of studies have found a medium to high correlation between children's pattern and structure competencies and their attainment in mathematical achievement tests (Lüken, Peter-Koop, & Kollhoff, 2014; Rittle-Johnson, Fyfe, Hofer, & Farran, 2017; Rittle-Johnson, Zippert, & Boice, 2019; Schmerold et al., 2017). (For a review of empirical studies on patterning see Burgoyne, Witteveen, Tolan, Malone, & Hulme, 2017; Pasnak, 2017). Furthermore, fostering patterning competencies in intervention studies not only showed a positive effect on children's pattern knowledge (Hendricks, Trueblood, & Pasnak, 2006; Papic, Mulligan, & Mitchelmore, 2011), but also on their arithmetical competencies (Kidd et al., 2013, 2014; Lüken & Kampmann, 2018). Children with lower mathematical abilities especially benefited from such interventions, i.e., teaching that made patterns and structures explicit, and strengthened children's abilities to consciously recognize patterns, interpret structure, and establish relationships.

However, although there is general agreement that pattern competencies are important for children's mathematical development, the foundations and development of these competencies in early childhood have not yet been systematically explored. Furthermore, most studies on pattern competencies exclusively look at the *product*, i.e., the solution, in order to measure children's understanding of patterns. In other mathematical domains (e.g., arithmetic), teachers and researchers additionally look at the *process* of reaching the solution. For instance, the task 6 + 7 can be solved correctly both by a counting strategy (e.g., count on: 7, 8, 9, 10, 11, 12, 13) or a doubling strategy (e.g., 6 + 6 + 1 = 13). The counting strategy would be seen as a more basic, and the doubling strategy

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as a more advanced strategy for solving additions (Thompson, 1999), because structural relationships between the numbers are used with the doubling strategy. Transferring this consideration to the domain of patterns, studies on patterning should take children's strategies, as well as their solutions into account when describing an understanding of patterns. As there are very few studies on children's patterning strategies, it is so far undetermined if young children show patterning strategies at different levels of sophistication, and if the use of patterning strategies develops in early childhood without formal instruction.

In the present study, we aim to add to the existing research on early patterning by examining both the children's strategies and solutions on a variety of patterning tasks. First, we look at the specific type of pattern – repeating patterns. Next, we summarize research on young children's performance of patterning tasks and their patterning strategies, and finally we describe the present, a mixed-method study design.

Mathematical patterns and patterning tasks

The mathematical patterns for early childhood education can be categorized into three types: spatial structure patterns, repeating patterns, and growing patterns (Papic & Mulligan, 2007). They differ by their structure, i.e., in the way in which the various elements that make up the pattern are organized and related. Because of their differing structure, different strategies are to be expected for each of the three mathematical patterns. The current research, therefore, focuses exclusively on one type of pattern. For repeating patterns, no numerical considerations are needed, which makes them accessible for very young children (Rittle-Johnson, Fyfe, McLean, & McEldoon, 2013). In fact, young children create repeating patterns spontaneously during their play (Ginsburg, Inoue, & Seo, 1999; Threlfall, 1999). Recent longitudinal and causal evidence indicates that not only patterning skills, which include all three types of patterns, are important to math development, but also repeating patterning skills alone are a unique predictor of later math achievement (Lüken et al., 2014; Nguyen et al., 2016; Rittle-Johnson et al., 2017). Hence, it seems relevant to look in more detail at young children's understanding of repeating patterns and how they solve repeating patterning tasks.

A repeating pattern is a periodic sequence of elements (e.g., ABCABCABC …) which can be reduced to a smallest subset that is called the *unit of repeat* (Threlfall, 1999). The unit of repeat is repeated in the form of a geometric translation and thus creates the repeating pattern. Repeating patterns have a cyclic structure (Liljedahl, 2004) and, hence, any unknown element of the sequence can be predicted (Mulligan & Mitchelmore, 2009; Wijns, Torbeyns, De Smedt, & Verschaffel, 2019b). A lot of mathematical topics in primary and secondary school are based on this same or a very similar structure, such as congruence mappings, counting in steps, multiplication as addition of equal parts, fraction as part-whole, decimal expansions of rational numbers or trigonometric functions (Lüken, in press).

The difficulty of a repeating pattern depends on the complexity of the repeating unit. AB patterns, in which the unit of repeat consists of two different elements that each occur only once, are considered the easiest. With the increase of the numbers of elements (e.g., ABC, ABCD) and the repetition of single elements (e.g., AAB, AABB) the pattern's difficulty increases (Clements & Sarama, 2009; Vitz & Todd, 1967, 1969).

Although the term *patterning* has been defined as "the search for mathematical regularities and structures" (Sarama & Clements, 2009, p. 319), in this study *patterning* refers exclusively to repeating patterns. Patterning abilities, therefore, are defined as children's competencies regarding repeating patterns; we understand patterning tasks as activities that are executed with repeating patterns.

In both early childhood settings and research studies on early patterning, a wide range of tasks are used to assess and foster children's abilities with repeating patterns (for a classification of patterning tasks see Wijns et al., 2019b). The most common tasks are *copy, extend*, and *repair* (Rittle-Johnson et al., 2013). Copying a pattern means reproducing a model pattern which is either in view or hidden. This task is also called *duplicate* in some literature (Clements & Sarama, 2009). Extending a pattern means finding the next element of a pattern and repairing is filling in missing elements of the pattern, which is also named *fix* (Clements & Sarama, 2009) or *interpolate* (Wijns et al., 2019b). McGarvey (2012) hints at the fact that these tasks emphasize "the recursive organization of elements rather than repeating units" (p. 334) in a repeating pattern. In this view, the pattern would be perceived as different relations between successive elements (e.g., for an AAB pattern: Next to A is A if previously was B. Next to A is B if previously was A. Next to B is A.). The periodicity of repeating patterns is emphasized. More complex tasks that are furthermore considered to help children to see the unit of repeat are *translate, identify the unit of repeat*, and *create*. In translating a pattern, one produces a pattern with the same structure using different materials. This task is called *generalize* (Wijns et al., 2019b) or *abstract* (Rittle-Johnson et al., 2013) in other research studies. Identifying the unit of repeat means to find the smallest part that produces the pattern, while creating a pattern would be to invent a pattern.

In the description above it becomes apparent that different patterning tasks might emphasize as well as challenge different types of thinking, namely *recursive* versus *functional* thinking (McGarvey, 2012; Wijns et al., 2019b). Relating to repeating patterns, recursive thinking would be to use the relationship between consecutive elements when predicting an unknown element. In this way, it is only possible to predict the next element in the sequence, and then the next, one element after the other, starting from the last known element (i.e., $a_{n+1} = f\,(a_n)$). In contrast, functional thinking in a repeating pattern context would be to identify the unit of repeat and to use the pattern's structure to predict any element of the sequence (i.e., $a_n = f\,(n)$; Wijns et al., 2019b). Helping students shift from recursive reasoning to perceiving a unit of repeat within a pattern is seen as an essential conceptual understanding (Economopoulos, 1998; McGarvey, 2012; Threlfall, 1999). To the best of our knowledge, no research has yet systematically examined young children's recursive versus functional approach to repeating patterns.

Research on young children's patterning competencies – with a focus on the solutions

Several studies have documented young children's performance on repeating patterning tasks (e.g., Papic et al., 2011; Rittle-Johnson, Fyfe, Loehr, & Miller, 2015; Rittle-Johnson et al., 2013; Rustigian, 1976). Findings indicate that young children can solve various repeating patterning tasks before school entry. As with other mathematical knowledge, repeating pattern competencies differ largely between and within age groups. Rustigian's (1976) cross-sectional data with 3-, 4-, and 5-year-old children show the lowest solution rates on selected tasks for the 3-year olds, and the highest solution rates for the 5-year olds. Rittle-Johnson et al. (2013), (2015) found large individual differences among 4-year olds' repeating pattern knowledge, ranging from only being able to copy a repeating pattern to being able to explicitly identify the pattern unit. Looking at mean solution rates, 77% of the 4-year olds were able to copy an AABB pattern, about half of the children could correctly extend patterns, and one-third were capable of translating repeating patterns. Fewer than 10% of the 4-year olds were able to identify the unit of repeat. Although total scores improved from fall to the following spring and substantial improvement was made in copying, extending and translating, children made no improvement in pattern unit recognition.

Sarama and Clements (2009) suggest a learning trajectory for repeating patterns, based mainly on several studies they conducted themselves. It is mostly in line with the results of the studies described above. According to this hypothetical developmental progression, 3-year olds are able to note regularities and recognize a simple repeating pattern, whereas 4-year olds can copy first AB, as well as later other simple patterns, and repair and extend AB patterns. 5-year olds are able to extend simple repeating patterns other than AB patterns, and 6-year olds to identify the smallest unit of the pattern.

Looking at the various patterning tasks, there is empirical evidence of different difficulty levels. Copying and repairing seem to be the easiest, next are extending and translating, and finally identifying the unit of repeat, which seems to be the most difficult task (Clements, Sarama, & Liu, 2008; Rittle-Johnson et al., 2013; Rustigian, 1976). Wijns, Verschaffel, De Smedt, and Torbeyns (2019c) showed that not all patterning tasks contribute to children's mathematical development. In their study, only translating and identifying the unit of repeat, but not extending were associated with numerical ability one year later. The associations were unique above age, prior numerical ability, spatial ability, and visuospatial working memory. This finding might be related to McGarvey's (2012) suggestion that some patterning tasks can be correctly solved with different conceptual understanding of repeating patterns. For extending a repeating pattern it is not necessary to abstract the unit of repeat. The extension can easily be found by rhythmically alternating the elements (Threlfall, 1999). In contrast, finding the smallest part that produces the pattern requires the child to abstract and apply the unit of repeat explicitly (Lüken, in press). An understanding of the pattern's structure, however, seems to be critical for children's mathematical development. This consideration is supported by empirical evidence of links between the application of repeating patterns' structure and both multiplicative reasoning (Mulligan, 2002) and spatial skills (Rittle-Johnson et al., 2019; Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2017). From a didactical point of view these links make sense in so far as for multiplication, an understanding of equal units that are made out of units – like in repeating patterns – is fundamental (Steffe, 1988). Further studies are needed on children's conceptual understanding of repeating patterns that manifest in different strategic approaches to patterning tasks. With regard to educational practice, a variety of patterning tasks should be examined.

Research on young children's patterning competencies – with a focus on the strategies

Patterning tasks can be solved in many different ways. Such ways are usually called *strategies* (Threlfall, 2009), following Ashcraft's (1990) definition of strategy as "any mental process or procedure in the stream of information-processing activities that serves a goal-related purpose" (p. 207). Children often possess multiple strategies for solving problems and select among them based on task demands, as well as their proficiency in executing the strategies available to them (Chen & Siegler, 2000; Siegler, 1996). In contrast to research on patterning solutions, there has been very little research done on patterning strategies.

By all accounts, Rustigian (1976) was the first to describe young children's patterning strategies, which she called "response techniques" (p. 189) in her dissertation on pattern recognition. She worked with a sample of n = 60 3- to 5-year-old children on repeating patterns. The children were asked to copy (with model pattern in view), identify (select a pattern with the same structure than the model pattern), and extend four different patterns. After a quantitative analysis of the data, categories which were generated from "the investigator's written record of techniques or strategies each subject was observed to employ" (p. 188) were described and quantified for each of the three tasks. Six different techniques were listed for both the tasks copy and extend. Both category lists started with a "random" and ended with a "correct placement technique" (p. 192 ff). In the categories in between, children's responses focused either exclusively on the similarities, or the similarities and differences between the pattern's elements. A developmental hierarchy for these response techniques was suggested by Rustigian (1976) based on the percentage of technique use for each age group.

In an attempt to describe developmental pathways to repeating pattern-making, Garrick (2000) conducted a longitudinal study (n = 24) with 3½- to 4½-year old children. In addition to assessing children's procedural skills in copying (model pattern in view and hidden), extending, and creating repeating patterns, she observed children's creation of patterns in the naturalistic setting of a nursery class. She described in detail how children created patterns by differently complex color organization. Similar to Rustigian (1976), the hierarchically ordered levels include strategies from random color organization, via grouping and sequencing colors based on similarity and difference to creating complex repeating patterns. Garrick (2000) highlights the awareness of color and the concept of classification, i.e., the purposive creation of similarities and differences by color as the very basic elements of pattern making. These elements were observed to appear relatively early in development and then led to more advanced strategies (Garrick, Threlfall, & Orton, 1999). A comparison of the longitudinal data (3½ years vs. 4½ years) on children's color organization levels revealed a progress for the majority of children (Garrick, 2000).

In the context of their extensive research on patterning, Mulligan and colleagues assessed n = 53 preschoolers' (from ages 3 years 9 months to 5 years) patterning skills as well as their pattern-related concepts (Papic et al., 2011). They were the first to not only describe strategies for single tasks, but in addition, built broader categories into which the strategies for a variety of tasks fell. The creation of categories and an in-depth description of strategies, however, were not reported. The tasks included copy (with and without model pattern in view), repair, create, and extend repeating patterns. The five strategy categories were listed in increasing order of sophistication and started, similar to Rustigian (1976) and Garrick (2000), with strategies where children chose and placed elements randomly (*random arrangement*). Strategies that matched items one-by-one were frequently observed and made up the category *direct comparison*. The most common strategy in this study was *alternation*, where children focused on the sequence of individual colors. For strategies in the fourth category, children were able to identify and use the unit of repeat (*basic unit of repeat*). Strategies where children demonstrated and expressed simple generalizations about the unit of repeat were sorted into the most sophisticated category *advanced unit of repeat*. A quantification of children's responses regarding the categories are partly reported in Papic et al.'s (2011) paper and show a general increase of more advanced strategies over the course of 18 months. Furthermore, children who took part in a patterning intervention showed even more advanced strategies than children from the control group.

In the fourth and most recent study on patterning strategies, Collins and Laski (2015) suggested that patterning tasks could be solved using either a one-to-one appearance matching strategy or a relational similarity strategy. For a one-to-one appearance matching strategy children would match superficial features without considering the pattern's underlying structure, for example, by copying a pattern by matching the color or shape of each item in the pattern, one at a time. In order to mentally represent, abstract, and manipulate the unit of repeat, relational similarity strategies would be required. Preschoolers' (n = 66; from ages 3 years 11 months to 5 years 7 months) patterning strategies on four patterning tasks were inferred from complex error analyses. Results indicate that preschoolers do indeed use both kinds of strategies, but that the frequency with which they use each strategy depends on task complexity. Children tended to use an appearance matching strategy on copy (with model pattern in view) and extend tasks and a relational similarity strategy on translate tasks. Performance on the repeating unit identification tasks was very low, therefore it remains unclear which strategy children prefer to use on this kind of task. Furthermore, Collins and Laski (2015) description of error coding reveal a third-strategy category, called *off-task errors*. Strategies here include placing elements randomly, sorting elements by color or shape, or using elements for building.

In summary, the studies reviewed above show that different strategic approaches to patterning tasks are possible and that young children use various strategies when solving patterning tasks. A random approach to repeating patterns, comparison strategies, and the use of the unit of repeat seem to be prominent approaches to patterning tasks, as they appear in most of the reviewed studies. There are preliminary empirical indications that strategy use develops with increasing age. In almost all studies, patterning strategies are touched on only briefly and as a by-product in the context of other research interests. Further studies are needed to systematically examine young children's strategies regarding a variety of patterning tasks and how these strategies are distributed among children of different age. As all reviewed studies were conducted in settings where patterning was part of early childhood curriculum, research on the organic development of patterning strategies is still needed.

The current study – overall research design

The current study aimed at systematically observing, describing, and measuring 3- to 5-year-old children's informal strategies when working on various repeating patterning tasks. Specifically, we sought to characterize young children's strategy choice across a range of patterning tasks and to examine the organic development of young children's understanding of repeating patterns in early childhood. A mixed methods research design was conducted to answer the following research questions:

- What kind of strategies do 3- to 5-year-old children use when solving different patterning tasks?
- How and to what extent does strategy choice change between the age of three and five?
- What factors influence patterning strategy choice?

We chose a sequential qualitative-quantitative design (QUAL → QUAN) (Morse, 2003) for the purpose of extending the breadth and range of inquiry ("expansion"; Greene, Caracelli, & Graham, 1989). As no model of patterning strategies and their development has been described in literature, a systematic, longitudinal survey of patterning strategies was needed to answer the first research question and develop hypotheses regarding the second. We therefore started with an exploratory study by observing six children's solution processes on a variety of patterning tasks over the course of 2 years (starting when the children were 3-years old). The results – the integration of strategies into five hierarchically ordered strategy categories and hypotheses regarding the development of strategy competencies in early childhood – were then tested quantitatively and cross-sectionally with children of three age groups. The larger sample was used to test the transferability of the findings from the small N study (first research question) and to describe the investigated phenomena on an aggregated level in order to get a general picture of the distribution of patterning strategies in early childhood (all research questions). The quantitative and the qualitative sub-studies fulfilled different, yet complementary, purposes within this mixed-method design: The qualitative methods were used to gain access to specific knowledge in the field in order to develop theoretical concepts. The findings then informed the quantitative sub-study. The quantitative methods gave an overview of the domain under study and described its heterogeneity. Thus, we used the triangulation of methods "as a mutual supplementation of research results which provides a more complete picture of the research domain." (Kelle & Buchholtz, 2015, p. 323)

The results of this study will bring greater clarity to how 3- to 5-year-old children solve patterning tasks and understand repeating patterns, which in turn is influential for later mathematical skills and achievement. The results might also have an influence on the teaching of repeating patterns.

Summary – qualitative sub-study I

The longitudinal, in-depth exploration of patterning strategies and the development of strategy categories with different levels of sophistication were the goals of the first sub-study. Due to space limitations, the qualitative sub-study I is only reported as a summary. For a detailed description of the sample, the case analyses, and the development of the superordinate categories, see Lüken (2018). As the quantitative sub-study II depends on the qualitative results, and the latter are fundamental for an understanding of the quantitative results, they are summarized in more detail.

Methods

Six children were followed during their first 3 years of kindergarten[1] (four girls, M_{age} at first interview = 3 years 6 months) and interviewed three times individually (18 interviews total), using a clinical, task-based interview format (Ginsburg, 1997). Six patterning tasks were administered during the interviews to observe the children's strategy use when working with repeating patterns: explain, copy with pattern in view, copy with pattern hidden, repair, extend, and translate (see Table 2 for detailed instructions). All tasks were conducted consecutively with three repeating patterns that differed according to their unit of repeat (AB, ABC and AAB pattern).

The interviews were analyzed by means of "qualitative content analysis" (Mayring, 2008). In an iterative process, combining deductive and inductive steps, all strategies were described, categories defined, and the children's strategy for each task was coded by the help of these categories. Strategies were then compared according to the level of attention to the pattern's structure, and integrated into five superordinate categories. Furthermore, single case analyses were conducted for each child. On this basis, hypotheses for the development of repeating patterning competencies were generated.

Results

The strategy categories that are presented in this section are the main results of the qualitative sub-study I. Still, the version that is described here is the end-outcome after the evaluation and revision of the categories during the analysis with the larger sample (sub-study II). The observed patterning strategies are described in a condensed way for each category. Some common examples are given (for more strategy examples see Table 1), and children's general view of repeating patterns is summarized for each category. The strategy categories are arranged hierarchically and reflect an increase of understanding of the regularities and order within a repeating pattern.

1 No reference to pattern besides reproduction of the pattern's gestalt All strategies which neither refer to the specific features of the elements nor to the regularity of the pattern are subsumed in this most basic strategy category. Children employ strategies that are based on guessing, personal preference, or random selection. Common examples include guessing that a red cube is missing in a green-yellow pattern when asked to fill in the gap, or choosing different colors or shapes than those

Table 1. Strategy categories with descriptions and examples of strategies.

Strategy category	Description	Examples of strategies
No reference to pattern besides reproduction of the pattern's gestalt	Use of other elements than given in the pattern AND no regularity at all in children's patterns	• Randomly selecting new elements • Selecting new elements for subjective reasons • Guessing impossible elements • Reproducing the pattern's gestalt
Attention to singular characteristics	Reference to singular aspects of the pattern (e.g., color, shape, number) AND generally no regularity in children's patterns	• Using the same elements but in an arbitrary or different order • Guessing possible elements • Repeating the last element (*extend; last element*) • Producing AB pattern when not appropriate (*translate*)
Comparison & classification	Comparison and classification of singular elements within or between patterns	• Comparing and classifying pattern's elements (*explain; translate*) • Matching elements in a one-to-one correspondence (*copy*) • Comparing and matching elements (*repair; translate*) • Looking at and comparing with the pattern's beginning (*extend; last element*)
Focus on sequence	Focus on the succession of elements; seeing a sequence of "what comes next"	• Alternating or cycling through the elements without discerning the pattern's structure • Focusing on the relation of successive items ("Next to orange is purple, next to purple is green, next to green is orange, next to orange is purple, … ") • Chanting rhythmically
View of unit of repeat	Identification and use of the repeating unit	• Taking all elements of the unit at once out of the box • Arranging all elements of the unit at once • Naming the unit of repeat and its repetition ("It's always red-blue.") • Naming and actively showing the unit of repeat ("Red-Blue!")

in the model pattern when copying or extending a pattern. Still, most children will arrange the pattern's objects in a line, thus recognizing the linear arrangement. Their general perception of a pattern seems to focus on the external shape of the pattern only. Two patterns are thus considered the same if they have the same gestalt.

2 Attention to singular characteristics Strategies of the second strategy category refer to singular, mostly external aspects of the pattern, in addition to its gestalt. They do not relate to the pattern's structure. For example, children either choose the same colors or the same shapes as in the pattern, they enumerate the pattern's elements, or they purposely recreate the same length. However, little regularity can be found in the children's patterns altogether. The children's general view of patterns seems to focus on the appearance or number of the pattern's elements. Two patterns are considered the same if they consist of the same elements; specific relations and regular order are not noticed.

3 Comparison & classification The first ideas of regularity become visible in the strategies of the third category. Children compare the pattern's elements, and highlight sameness within and between patterns on a basic, very concrete level. For example, same colors are classified within a pattern ("The yellows are the same.") or numbers of same colors are compared between two patterns ("*Three* purples and *six* blacks here. *Three* yellows and *six* oranges here."). A common strategy when copying a pattern is to execute a one-to-one correspondence. To look at the pattern's beginning, and to compare and match the extension step by step with the beginning, is a typical extension-strategy of this category. This procedure shows an emerging sense of some kind of regularity within the pattern, although the specific regularity – the pattern's structure – is not yet graspable for the child.

4 Focus on sequence A growing awareness of regularity within the pattern can be observed in the strategies that belong to the fourth strategy category. The strategies focus on the relationship between successive elements of the pattern. Children look at and describe the physical relation between adjacent elements, however, without explicitly referring to the repeating unit (e.g., "Next to green is purple, next to purple is orange, next to orange is green, next to green is purple, next to purple …"). Typical strategies for this category are alternating colors or cycling through the elements of the pattern over and over again, even chanting them rhythmically. The strategies of this category reflect recursive thinking. A pattern appears to be a sequence of "What comes next." The children seem to be aware that the elements are ordered in a regular way, without explicitly grasping the pattern's structure. The elements of the pattern are rather seen as strung together as on a chain. Children are not yet able to break the pattern down into its units of repeat.

5 View of unit of repeat In the most advanced strategy category, the strategies show the children's understanding of the pattern's structure and, therefore, reflect functional thinking. They explicitly name or show the repeating unit. A common strategy for younger children is to take all elements that make up the unit of repeat at once out of the box or group them together before arranging them in a line. The children know that there is a smallest part that produces the sequence – they are able to identify this unit of repeat and use it during the tasks.

The strategies of categories 3–5 can result in correct solutions (but not always do); solutions realized by strategies of categories 1 and 2 are almost always wrong. Table 1 gives a condensed overview and description for each strategy category.

A quantification of the six children's strategies for every measuring point revealed that the 3-year olds only used strategies from categories 1 and 2. The 4-year olds' strategies were scattered from category 1–4, and the 5-year olds' from 2 to 4 with the core area being categories 3 and 4. Thus, a development of strategies during the kindergarten years from basic to more advanced strategies was hypothesized. It was striking that a strategy from category 5 (view of unit of repeat) was used only by one child when copying the hidden AB pattern. In line with Radford (2012), and Sarama and Clements (2009) we hypothesized that the ability to recognize the unit of repeat may not develop organically until the children are older.

With the aim of quantitatively evaluating the model of strategy categories and testing the hypotheses regarding the development of strategy use in early childhood, the second, quantitative sub-study was planned and conducted.

Method – quantitative sub-study II

Participants

Consent was obtained for 174 children attending 15 kindergartens[1] in a metropolitan area in the northwestern part of Germany. Kindergartens were chosen from areas with different mean incomes to get a diverse sample regarding children's socio-economic background. The sample consisted of 54 children aged 3 (30 girls, M_{age} = 3 years 6 months, *SD* = 2.7 months, range = 2 years 11 months – 3 years 11 months, 76% speaking German as family language), 65 children aged 4 (33 girls, M_{age} = 4 years 5 months, *SD* = 3.2 months, range = 4 years 0 months – 4 years 11 months, 82% speaking German as family language) and 55 children aged 5 (22 girls, M_{age} = 5 years 5 months, *SD* = 3.6 months, range = 5 years 0 months – 5 years 11 months, 80% speaking German as family language).

None of the participating kindergartens were using a specialized curriculum focused on patterning, and teachers reported doing no repeating patterning activities at all (which we consider representative of German kindergartens). Therefore, it might be suggested that the findings of this study shed light on children's informal patterning knowledge and its organic development.

Design, materials, and procedure

For data collection, we used the same, task-based interview format as in the qualitative sub-study I. Each child worked one on one with a researcher in a quiet room in their kindergarten. The session took 45 min on average and was split over two days if the child showed signs of fatigue. In the interview, children completed eight patterning tasks with three different patterns each, followed by ten items on early numerical abilities (the latter are not applicable to this article). Throughout the interview, the interviewer asked the child how they found each solution and encouraged them to verbalize and show the way of finding the solution. Interviews were conducted by five members of the research team. The interviews were videotaped so that the interviewer did not need to take any notes, giving her the freedom to completely engage with the child. In addition, all actions, gestures, and the exact wording were available for the analyses.

Eight patterning tasks were designed to observe children's strategy use when working with repeating patterns (see Table 2). They were based on items that are common knowledge in mathematics education (e.g., Burton, 1982; Sarama & Clements, 2009) and are also published in other research studies (e.g., Papic et al., 2011; Rittle-Johnson et al., 2013). The tasks included describing model patterns (*explain*), duplicating visible and hidden model patterns by making an

Table 2. Description of patterning tasks; assignment of patterning tasks to sub-study.

Task	Instruction	Inclusion in sub-study
Explain	"Please, tell me about the pattern." "What's the same? What's different?"	I II
Copy (*the model pattern is in view while child responds*)	"Create the same pattern as mine."	I II
Copy (*the model pattern is hidden while child responds*)	"Create the same pattern as mine."	I II
Repair	"A cube is missing. What color is the missing cube?"	I II
Extend	"What comes next?"	I II
Name the last element	[*The pattern is extended by the interviewer with 3 (AB), 4 (ABC), 5 (ABCC) uncolored cubes.*] "Look, these cubes have lost their color. Imagine we re-color them according to the pattern. What color would be this last cube?"	II
Translate	"Use these counters [*different material and colors*] to create the same pattern."	I II
Identify the unit of repeat	"Cut the pattern into parts that are the same."	II

exact replica (*copy*), filling in a gap in the patterns (*repair*), extending patterns by at least two full units of repeat (*extend*), and recreating model patterns with different materials (*translate*). For the quantitative sub-study II, this six tasks were supplemented by two additional tasks in order to broaden the variety of patterning tasks. For the task *name the last element* a model pattern was extended by the interviewer in an "invisible" way by adding uncolored cubes. The children were asked to determine the color of the last, uncolored cube. The number of added cubes (see Table 2) was chosen in a way that the color of the last visible cube differed from the cube that was to be named (length of repeating unit +1). For the task *identify the unit of repeat* the children were invited to divide the pattern into its units of repeat (for similar tasks see Papic et al., 2011; Rittle-Johnson et al., 2013). The model patterns were made of three units of repeat so that it was not possible for the child to identify two equal repeating units that were non-minimal. Because pilot data suggested that children needed extra scaffolding on the unit identification task in order to understand what was being asked of them, children were given feedback on their first attempt and, if needed, they executed a second attempt, which was being scored. All tasks were tested in several pre-studies. The patterning tasks, together with the detailed instructions, are listed in Table 2 for both sub-studies, organized by the order in which they were administered. The same order was used for every child, because previous research by Rittle-Johnson et al. (2013) showed no significant differences in accuracy based on the order of task presentation.

All tasks were conducted consecutively with three repeating patterns that differed according to their unit of repeat. We used the same patterns and materials as in sub-study I except for the third pattern. To increase difficulty, we replaced the AAB by an ABCC pattern. The repeating units now contained two (AB), three (ABC) and four elements (ABCC), with only three elements being different in the last, four-element unit of repeat. As shown in Figure 1, the patterns were presented with three (AB) and two (ABC and ABCC) instances of the repeating unit. Whereas the ABCC pattern ended with an additional partial unit of repeat, the first two ended with a complete unit of repeat. All three patterns were constructed with colored wooden cubes, choosing the dimension of color over the dimension of shape. Since 3-year olds were interviewed, we assumed it easier to communicate about color than relying on children's knowledge of names for different shapes. To make sure that every child was able to distinguish the colors, the interviewer showed each colored cube and asked the child to name the color. Furthermore, during the interview the children were invited to not only name a color, but to also point at the respective colored cube. No children needed to be excluded because of color discrimination problems. Numerous cubes in six different colors were available for the children to choose from during all tasks. The interviewer used uncolored cubes for the task in which the last element had to be named. For the translation task, children's materials were counters in four colors that differed from the cubes' colors. Children's material for the identification task was a plastic knife.

Analysis

Each child's answer to every task with all three patterns was coded for strategy (i.e., 24 answers for each child) and correctness (i.e., 21 answers for each child, see below), using the qualitative data

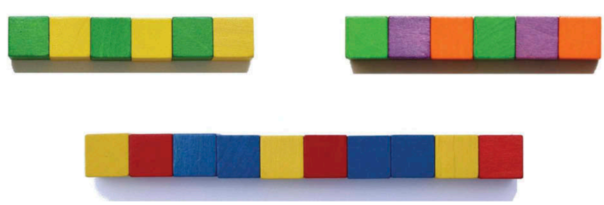

Figure 1. Patterns of sub-study II: AB pattern (green, yellow), ABC pattern (green, purple, orange) (*top*) and ABCC pattern (yellow, red, blue, blue) (*bottom*).

analysis software MAXQDA 12 (VERBI, 2016). The coding manual that had been developed in the qualitative sub-study I (for an abbreviated version see Table 1) was used to assign each strategy to the corresponding category. Children's actions, gestures, and their exact wording were taken into consideration while coding. To code children who showed multiple strategies, we used competency-oriented coding, which assigned the highest category observed for the specific task.

The coding manual also contained coding instructions regarding the correctness of a solution. The correct solution for the tasks *repair* and *name the last element* were unambiguous, as only one correct color was possible. Both *copy* tasks and the task *translate* were coded as correct if a pattern with the same structure was produced, even if the colors and/or the total length differed. The responses to the task *extend* were considered correct if the child produced at least two full repeating units of the pattern and made no errors. If a child extended the pattern by a symmetric mirror image the solution was also coded as correct, because at no point of the interview we had explicitly defined our understanding of pattern as a translation congruence mapping. On the *identify repeating unit* task, the child had to separate the pattern into the respective three units of repeat. We coded the task *explain* only for strategy, as it was an open question that cannot be considered as right or wrong. To assess reliability, 20% of the answers were coded double. The inter-rater reliability calculated as the percentage of answers which received the same code from both coders was high (97% agreement).

A total score for each child was summed up for the correct pattern solution, referred to as *patterning skills score* hereafter. We also generated a *patterning strategy score* by adding up the level of strategy category (1–5) for each answer. For patterning skills score and patterning strategy score, linear regression analysis was used to model their relationship with age, adjusted for gender. Moreover, to model the association between patterning skills score and patterning strategy score we added patterning strategy score to age and gender as a predictor of patterning skills score.

For the description of the data in graph and table, age was categorized in three groups, whereas the age was kept continuous in the regression analysis to allow for a better modeling of the relationship between the outcomes and age.

Results – quantitative sub-study II

Performance on patterning tasks

The relative frequencies of correct solutions for the patterning tasks are given in Table 3. The tasks are arranged by difficulty, based on the 4-year olds' performance on the AB and ABC pattern. Although slightly differing in detail according to age and pattern type, tasks can be assigned to three difficulty levels based on the percentage of correct solutions for all age groups (easy: *repair, copy with model pattern in view*; medium: *extend, copy from memory*; difficult: *name the last element, translate, identify repeating unit*). Results for the three different pattern types show that the AB pattern is the easiest, the ABCC pattern the most difficult pattern for all age groups.

Table 3. Relative frequencies (%) of correct solutions for patterning tasks.

	3-year olds			4-year olds			5-year olds		
%	AB	ABC	ABCC	AB	ABC	ABCC	AB	ABC	ABCC
Repair	42.6	7.4	3.7	78.5	63.1	24.6	98.2	92.7	87.3
Copy (pattern in view)	18.5	9.3	3.7	73.8	60.0	27.7	92.7	96.4	85.5
Extend	20.4	5.6	0	66.2	29.2	6.2	94.5	65.5	41.8
Copy (pattern hidden)	14.8	3.7	1.9	66.2	23.1	10.8	94.5	61.8	47.3
Name last element	3.7	0	0	43.1	12.3	3.1	54.4	40.0	30.9
Translate	3.7	0	0	26.2	10.8	1.5	67.3	40.0	21.8
Identify repeating unit	0	1.9	0	10.8	6.2	3.1	9.1	9.1	7.3

Table 4. Regression coefficients for the association between age and gender and the outcomes number of correctly achieved patterning tasks (patterning skills score) and pattering strategy score.

Outcome	age (month) (95% CI, *p*-value)		gender (girl) (95% CI, *p*-value)	
Patterning skills score	0.55 [0.45, 0.62]	<0.001	1.83 [0.67, 3.00]	<0.01
Patterning strategy score	1.95 [1.72, 2.18]	<0.001	6.23 [2.17, 10.20]	<0.01

The patterning skills score had values ranging from 0 to 21 points, with a mean of 6.76 points (SD = 5.73). Resulting from the linear-regression analysis, age is seen as a major determinant of success in correctly solving patterning tasks, explaining 59% of the variability in the data. An increase of 1 month leads to an increase of 0.55 (95% confidence interval [0.49, 0.63]) in the number of correctly achieved patterning tasks; see Table 4 for details). Gender plays a statistically significant role in solving patterning tasks correctly with girls achieving on average 1.85 (95% confidence interval [0.67, 3.00]) more points in the patterning skills score (see Table 4).

Patterning strategies

Strategies from all five categories have been deployed in all age groups, albeit, in strongly varying frequencies (see Table 5). Figure 2 provides a summary of the strategy use regarding all three patterns for the age groups (see also Table 5, row 1). For the 3-year olds, the most frequently used strategies are from category 1 (81%), strategies from category 2 hold a proportion of 11%, the others are 5% (category 3) and under. The distribution for the 4-year olds differs quite strongly from the 3-year olds.' The most frequently used strategies are still from category 1 (50%), but strategy categories 2–4 share nearly equal parts (19%, 16%, and 13%, respectively). Strategies from category 5 are only used in 2% of all tasks. For the 5-year olds, the use of strategies from category 1 has further decreased (12%); the most frequently used strategies are from the categories 3 and 4 (both 32%). Still, strategies from category 5 are used least frequently (8%).

In contrast to the most condensed presentation in Figures 2, Figure 3 displays the most detailed view of the strategy data: the relative frequencies of strategies for each patterning task with each pattern type. Two patterns can be detected in the data. First, when the pattern type was more difficult (comparison between groups of bars from left to right), children more frequently used

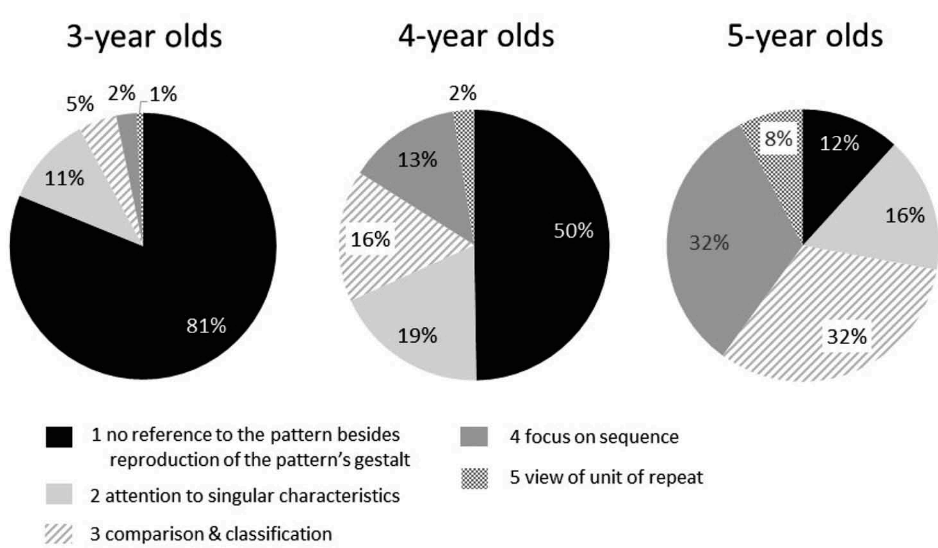

Figure 2. Relative frequencies of patterning strategy categories in total per age group.

Table 5. Distribution of frequencies (%) of strategy categories for 3-/4-/5-year olds.

%	**1** no reference to pattern besides reproduction of the pattern's gestalt	**2** attention to singular characteristics	**3** comparison & classification	**4** focus on sequence	**5** view of unit of repeat
Overall	81/50/12	11/19/16	5/16/32	2/13/32	1/2/8
AB	57/25/6	26/25/14	10/20/27	6/27/45	1/3/8
ABC	89/47/10	6/21/19	3/18/34	1/10/28	1/3/9
ABCC	97/77/20	1/10/16	1/9/35	1/3/23	0/1/5

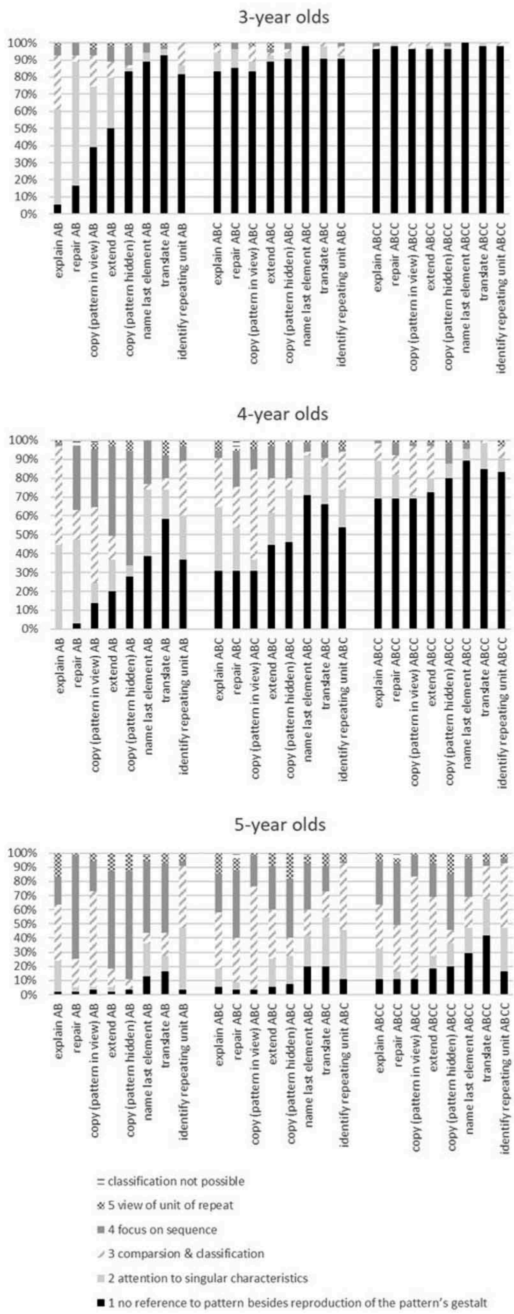

Figure 3. Relative frequencies of patterning strategy categories regarding each task with each pattern for the 3-year olds (*top*), 4-year olds (*middle*), and 5-year olds (*bottom*).

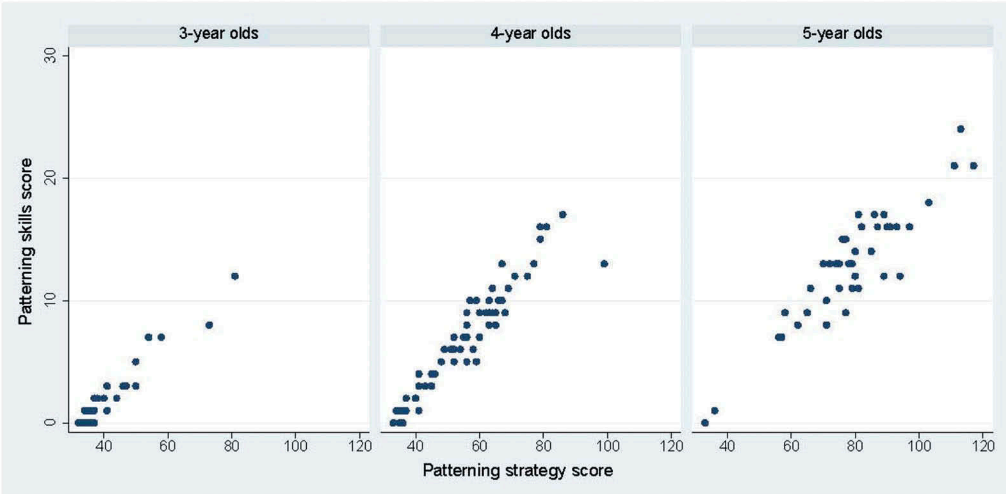

Figure 4. Relationship between patterning strategy score and patterning skills score per age group.

strategies from lower categories (see also Table 5). Second, when the task was more difficult (comparison within group of bars from left to right), children more frequently used strategies from lower categories. Both findings hold true for all age groups.

A third pattern we found was that older children used a higher number of strategies. The number of used strategies from different strategy categories was 2.7 (*SD*= 0.88; min = 1, max = 5) for the 3-year-old children, 3.8 (*SD* = 0.91; min = 2, max = 5) for the 4-year olds and 4.2 (*SD* = 0.79; min = 2, max = 5) for the 5-year-old children.

The strategy score had values ranging from 32 to 117 points, with a mean of 55 points (*SD* = 21 points), and with 25% of all values under 36. Resulting from the linear regression analysis, the strategy score increases every month by almost 2 points (95% confidence interval [1.72, 2.18]) on average with girls having a score higher by 6 points (95% confidence interval [2.17, 10.20]) than the boys (see Table 4). The patterning strategy score was highly correlated with the patterning skills score: including the patterning strategy score as predictor of the patterning skills score to the afore mentioned model yield a R^2 value of 94% and the co-variate age and gender were no more significant. That the relationship between patterning strategy score and patterning skills score is not age-dependent is illustrated in Figure 4.

Discussion

In line with contemporary mathematics education, which considers children's solution processes as well as their products, this study systematically examined 3- to 5-year-old children's strategies when solving repeating pattern tasks.

Patterning strategies

Results from both sub-studies show that young children use a variety of different patterning strategies. These patterning strategies can be rated as more or less sophisticated with regard to the extent in which they refer to the pattern's structure. Integrating patterning strategies for different tasks into superordinate categories led to five strategy categories which each reflect a different understanding (or maybe just a different view) of repeating patterns. The present model extends findings from earlier studies (Collins & Laski, 2015; Garrick, 2000; Papic et al., 2011; Rustigian, 1976). In the work with very young children (i.e., 3-year olds and young 4-year olds), it was important to further differentiate basic strategies, that mostly result in wrong solutions (Collins & Laski, 2015; Papic et al., 2011), into two categories. In the first, most basic category (1 *no reference to pattern besides reproduction of the pattern's gestalt*), strategies like guessing or choosing elements randomly still lead to children reproducing the linear arrangement of the repeating patterns and thus recognizing a very basic feature of this particular kind of pattern. When looking at the specific elements that are chosen, the random and guessing strategies need to be subclassified: Some children choose different colors than in the model patterns whereas others very consciously only use the ones that are given. Recognizing the specific color (for other patterns it might be the specific shape) as a relevant characteristic of the pattern seems to be an important developmental step (2 *attention to singular characteristics*). From the third category and onwards, strategies not only take the external characteristics but increasingly the pattern's regularity into account. Strategies that compare and match pattern's elements feature in previous studies (Collins & Laski, 2015; Papic et al., 2011; Rustigian, 1976) and constitute the third category (3 *comparison & classification*). It can be inferred from the strategies that the children recognize the orderly arrangement of the pattern's elements without grasping (or attending to) the specific regularity. Category 4 (*focus on sequence*) combines strategies that reflect a recursive approach in which children focus on the relation between adjacent elements (McGarvey, 2012). Breaking a pattern down into its units of repeat is considered a relevant mathematical competence (see, e.g.,, Threlfall, 1999). Similar to Papic et al. (2011), we observed children exhibit strategies that explicitly referred to or used the pattern's unit of repeat and its repetition in finding a solution (5 *view of unit of repeat*). The construction of these patterning strategy categories might help to better understand children's patterning competencies. Similar to a learning trajectory, the classification of a child's strategies could be a starting point when planning to foster their competencies.

The quantification of the strategy categories shows a variability of strategy choice: The same child uses patterning strategies from different categories for solving different tasks or for solving the same task with different repeating patterns. The average number of employed patterning strategies increases with age. Furthermore, the finely fanned out distribution of strategies for each task with each pattern reveals an interesting pattern of results: Independent of age, children generally use less sophisticated strategies with more difficult tasks, as well as patterns, and they use more sophisticated strategies with less difficult patterns and tasks. Similar strategy variability has been found in other mathematics tasks, such as arithmetic. Studies of children's addition strategies indicate, for example, that the same child will retrieve an answer from memory for some problems and use a counting strategy on others (Siegler, 1987). Problem difficulty and individuals' proficiency have been found to influence children's strategy choice in arithmetic (Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Kerkman & Siegler, 1993; Siegler, 1996). The present pattern of results suggests that children's choice of patterning strategy is also influenced by problem difficulty (here: tasks difficulty as well as pattern difficulty) and individuals' proficiency.

Patterning skills

The results on children's patterning skills (correct solutions) are in line with findings from previous research. For all three investigated age groups the difficulty of a repeating pattern, measured by the percentage of correct answers, increases with the length of the unit of repeat (Clements & Sarama, 2009; Vitz & Todd, 1967, 1969). Regarding patterning task difficulty, this study also confirms previous research (e.g., Clements et al., 2008; Rittle-Johnson et al., 2013). Furthermore, this study is the first that systematically tested eight different patterning tasks. The results suggest that specific task difficulty depends on pattern difficulty as well as the child's age. Averaged over all age groups and patterns, the patterning tasks can be assigned to three difficulty levels. The two easiest tasks are *repair* and *copy with model pattern in view*. The two tasks *extend* and *copy from memory* have a medium difficulty level. The three most difficult tasks are *name the last element, translate*, and *identify repeating unit*, whereas the last is also by far the most difficult task.

With the data of 4- and 5-year-old children's relative frequencies of correct solutions for patterning tasks, this study yields similar results to previous findings (Papic et al., 2011; Rittle-Johnson et al., 2013; Rustigian, 1976). Furthermore, it adds to existing research with empirical data of 3-year olds' patterning skills. If at all, 3-year olds seem capable of handling simple AB patterns, with very low correct solution rates for the more difficult patterns (under 10%). Still, only half of the 3-year olds are able to repair an AB pattern, and a fifth or less can extend and copy it. In contrast, nearly all 5-year olds solve these four tasks correctly.

Gender differences in patterning

Surprisingly, we found significant gender differences in both patterning strategies and patterning skills, with the girls outperforming the boys – although we found no gender differences in the numerical abilities for this same sample (not reported in this article). The latter, gender equality regarding early numerical competencies, has been shown before (Bakker, Torbeyns, Wijns, Verschaffel, & Smedt, 2019). To the best of our knowledge, none of the previous studies on early patterning have investigated the existence of gender differences regarding repeating patterning competencies. However, our findings are in line with preliminary findings from the first data collection of a current longitudinal study on the development and stimulation of early patterning (Wijns, Torbeyns, Bakker, De Smedt, & Verschaffel, 2019a). The data that has been reported so far shows that 4-year-old Flemish girls outperform boys on three tasks regarding repeating and growing patterns (Bakker, Torbeyns, Wijns, Verschaffel, & Smedt, 2018, October). A possible explanation for this gender difference in early patterning might be that girls draw from more experience because they possibly engage more often in pattern making during free play. Future studies are needed to prove if this really is the case.

Development of patterning competencies in early childhood

The hypothesized distribution of strategy categories for the different age groups, and with it a development in strategy use from more basic strategies with the 3-year olds to more advanced strategies from categories 3 and 4 for the 5-year olds, was confirmed by the quantitative sub-study II. The linear-regression model for the association of age and patterning strategy score shows that a child who is one year older has an increased strategy score by 23.4 in comparison to a younger child. This equals an increase of one strategy category for each answer for oneyear of age. This reflects a quite substantial development of patterning strategy competencies during early childhood.

Furthermore, in contrast to the small N qualitative sub-study I, sub-study II shows that strategies from all five categories are used by children from each of the three age groups. In particular, there are 3-year-old children who show attention to the pattern's structure, as well as 5-year olds who use strategies from the most basic category. However, the distribution of frequency is very different for the 3- to 5-year olds. The 3-year olds' overall strategies are mainly based on an understanding of the pattern's gestalt and, to a lesser extent, also on the perception of singular, superficial characteristics of the pattern, like color. This understanding is the same for half of the 4-year olds. The other half seems to have developed a basic awareness of regularity in the pattern (e.g., "The greens are the same, the yellows are the same.") or an attention regarding the succession of the specific colors (e.g., "Next to green is yellow, next to yellow is green, next to green is yellow, … "). It is interesting that the 4-year olds' strategies are distributed nearly evenly among the first four strategy categories for the AB pattern, thus showing a broad developmental range. The 5-year olds display an emergent understanding of regularity, order, and succession. Still, fewer than 10% of all strategies used by the 5-year olds hint at a recognition of the unit of repeat.

Partly in line with Threlfall (1999) and Radford (2012), this study found that some children refer to the repeating pattern's structure on their own accord; even children as young as three years old are capable of recognizing structure. The majority of children, however, have not yet developed this ability. Consequently, perceiving the unit of repeat may occur organically but, at least with very young children, it most often does not. Sarama and Clements (2009) noted that being able to recognize the unit of repeat may not develop until the age of six. Since formal schooling in Germany starts when children are six years old and patterns are part of the curriculum, it would be interesting to see the extent that age and school education on patterning fosters the recognition of the pattern's structure. It would also be worth while to compare 3- to 5-year-old children's strategy use in countries where formal schooling starts earlier and/or patterning is part of the preschool curriculum.

This study is the first to show the large development regarding patterning strategies which takes place between the ages of three and five. The older the child, the more the strategies the child uses are focused on regularity and structure. However, the strategy categories do not constitute a developmental stage model. The cross-sectional nature of the quantitative study means that only trends between groups or relationships between age and strategies, and not a development at individual level can be observed. Consequently, we cannot show if every child's patterning strategies will progress through all five categories.

This study also shows a large development in patterning skills (correct solutions). A one-year increase in age leads to solving 6.6 patterning tasks more on average, which is nearly a third of all patterning tasks (see linear regression analysis). This increase in skills leads to an increase in strategy score and the relation between the two remains unchanged with increasing age. That is, more correct solutions are related to the use of more sophisticated strategies which is not due to practiced patterning activities in the kindergarten setting. However, we have no information about home activities or spontaneous patterning activities such as making repeating patterns with blocks during free play (e.g., Ginsburg et al., 1999). Thus, children's development in patterning competencies may partly be explained by experience which increases with age. Other developmental mechanisms which might underlie the improving knowledge of repeating patterns might be increasing working-memory capacity. Previous studies have shown that working memory is particularly important for helping preschoolers identify, re-create, and learn about patterns (Miller, Rittle-Johnson, Loehr, & Fyfe, 2016; Rittle-Johnson et al., 2013). Increases in working memory capacity are thought to allow young children to transition from focusing on singular aspects of a task to coordinating attention to two dimensions (Case & Okatomo, 1996). During early childhood, general cognitive abilities are improving, which may support patterning competencies; for example, children's ability to encode and analyze information (e.g., Siegler & Chen, 1998). This ability should help children notice key features of the objects and how those features vary systematically. Another relevant cognitive skill developing in early childhood is relational thinking (Miller et al., 2016). To perceive the repeating pattern's structure, children must think about the relations between the elements of the pattern, rather than simply noticing the perceptual features of individual elements. Furthermore, children between the age of three and five undergo a significant development in their language ability. Children's explanations form the basis of categorization and with more elaborate language it is more likely that the interpretation of a child's strategy reflects their true thinking.

Educational implications

There are several potential instructional implications that arise from this work. First, one should consider children's strategies, as well as their solutions, when doing patterning activities because a correct solution does not imply a child's understanding of the pattern's structure. Even the task *translate* (recreating a model pattern with different materials), which is promoted as helpful for children to grasp the pattern's underlying structure (Hoenisch & Niggemeyer, 2004; Warren & Cooper, 2006), can be solved correctly without referring to the structure by strategies from category 3, i.e., comparing and matching strategies. Thus, it seems important for educational settings to ask children how they reached their solution, letting them explain their thinking, and talking together about different approaches to solving repeating pattern tasks.

Results also reveal that only few 3- to 5-year-olds informally develop an understanding of a repeating pattern's structure. It seems that instruction is needed to guide children past basic procedural knowledge and recursive thinking about patterns toward a more conceptual understanding and a perception of the unit of repeat and its repetition.

Another instructional implication concerns the selection of appropriate patterning tasks. The distribution of strategies for the individual tasks shows that specific tasks elicit strategies, which are more basic than children would use on other tasks. For example, children using advanced strategies, or even showing an understanding of the pattern's structure, regress to doing a one-to-one-correspondence when copying a pattern with the model pattern in view. A task that has been proven to be helpful for making the repeating unit explicit is the task *identify*, in which a repeating pattern has to be broken down into its repeating units. Feedback from the interviewers in our research group, as well as hints within the context of other studies (Collins & Laski, 2015), suggest that this task is difficult to understand for children in an interview. However, many didactical ways could be developed in educational settings to use this activity in order to make the repeating unit explicit.

Learning about patterns is foundational to mathematical thinking (Charles, 2005; Kidd et al., 2013; Sarama & Clements, 2009). Future research should, therefore, test if patterning instruction, which takes strategies into account and explicitly focuses on repeating patterns' structure, promotes attention to rules and relationships that are transferable to other mathematical topics. More broadly, additional research is needed on to what extent the current findings generalize to patterning strategies with spatial and growing patterns and to older children.

Conclusion

As mathematics is the science of patterns and structures, repeating patterning activities are included in early education classrooms with the intention of supporting children's understanding of a central component of mathematics knowledge: the structure of mathematical objects. In German early childhood settings, where guided instruction on repeating patterns is not part of the curriculum, most children seem to not abstract the repeating unit, but instead to focus on single elements of the pattern and either classify or alternate them. However, for an understanding of number and arithmetical operations – especially the ability to conceptually subitize, a part-whole understanding, and an understanding of multiplication – it has been suggested that a change of focus from single elements to groups of elements, or put differently, creating units which are made out of units, is fundamental (Hunting, 2003; Schöner & Benz, 2017; Steffe, 1988). This importance of a functional, rather than a recursive view of mathematical objects, also generalizes to an understanding of growing patterns, algebra, and functions (Warren & Cooper, 2006; Zazkis & Liljedahl, 2002). The results of this study, therefore, bring greater awareness to the fact that the ability to discern and generalize patterns and mathematical structure develops spontaneously in only a small percentage of students before they enter formal schooling. Consequently, the role of a pattern and structure-based mathematical instruction is strengthened.

Note

1. German kindergarten comprises the 3 years before school entry, i.e., children start kindergarten when they are 3-years old.

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